



Interest Rate Derivatives: HJM and LMM

Chapter 31

HJM Model: Notation

$P(t, T)$: price at time t of a discount bond with principal of \$1 maturing at T

W_t : vector of past and present values of interest rates and bond prices at time t that are relevant for determining bond price volatilities at that time

$v(t, T, W_t)$: volatility of $P(t, T)$

Notation continued

$f(t, T_1, T_2)$: forward rate as seen at t for the period between T_1 and T_2

$F(t, T)$: instantaneous forward rate as seen at t for a contract maturing at T

$r(t)$: short-term risk-free interest rate at t

$dz(t)$: Wiener process driving term structure movements

Modeling Bond Prices

(Equation 31.1, page 712)

$$dP(t, T) = r(t)P(t, T)dt + v(t, T, \Omega_t)P(t, T)dZ(t)$$

We can choose any v function providing

$$v(t, t, \Omega_t) = 0 \quad \text{for all } t$$

Because

$$f(t, T, T_2) = \frac{\ln[P(t, T_1)] - \ln[P(t, T_2)]}{T_2 - T_1}$$

we can use Ito's lemma to determine the process for $f(t, T, T_2)$. Letting T_2 approach T_1 we get a process for $F(t, T)$

The process for $F(t, T)$

Equation 31.4 and 31.5, page 713)

$$dF(t, T) = \alpha(t, T, \Omega_t) \sigma_T(t, T, \Omega_t) dt + \sigma_T(t, T, \Omega_t) dz(t)$$

This result means that if we write

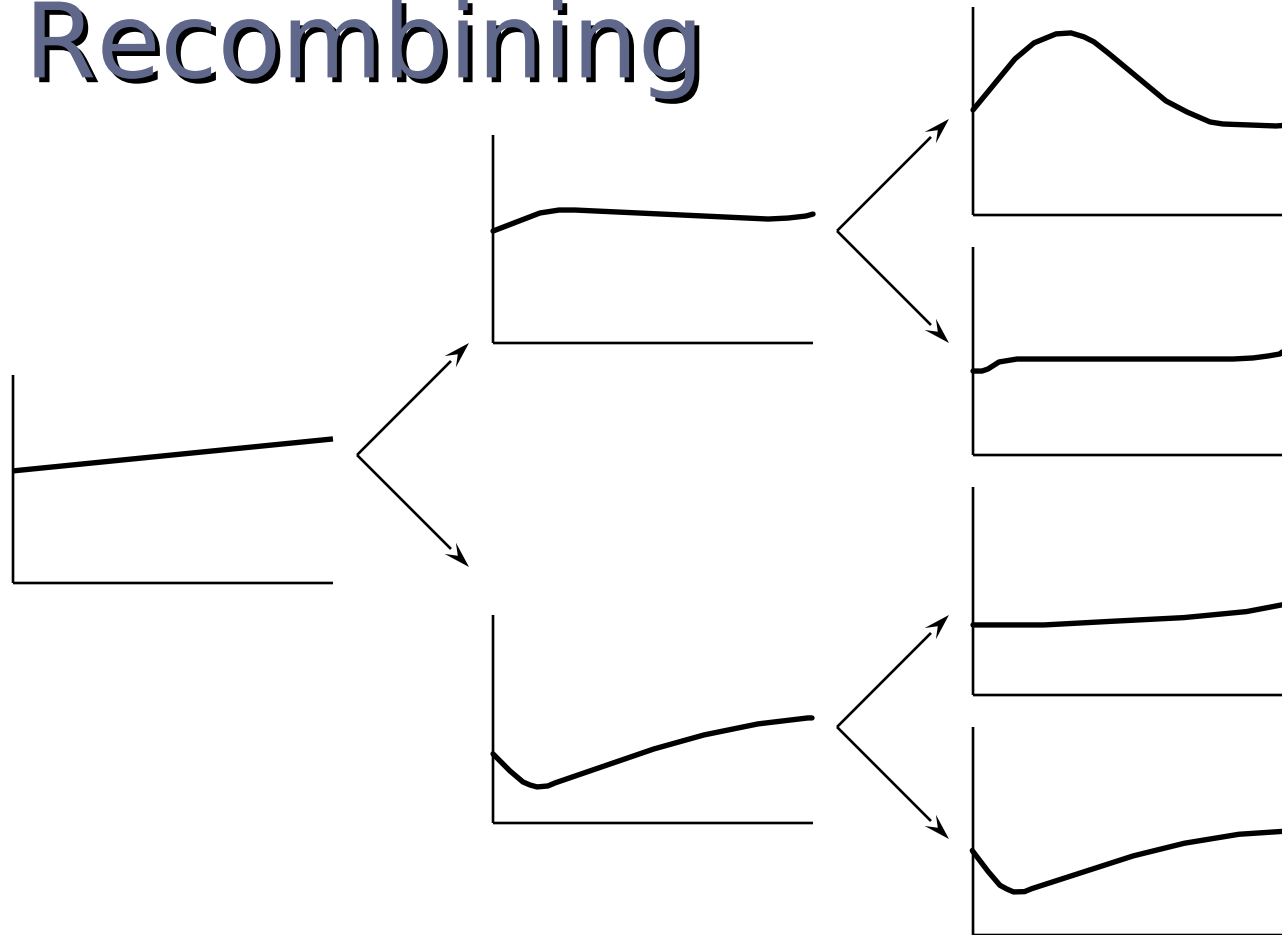
$$dF(t, T) = m(t, T, \Omega_t) dt + s(t, T, \Omega_t) dz$$

we must have

$$m(t, T, \Omega_t) = s(t, T, \Omega_t) \int_t^T s(\tau, T, \Omega_t) d\tau$$

Similar result holds when there is more than one factor

Free Evolution of Term Structure is Non-Recombining



Tree for the short rate r is non-Markov (see Figure 31.1, page 714)

The LIBOR Market Model

The LIBOR market model is a model constructed in terms of the forward rates underlying caplet prices

Notation

t_k : k th reset date

$F_k(t)$: forward rate between times t_k and t_{k+1}

$m(t)$: index for next reset date at time t

$\varsigma_k(t)$: volatility of $F_k(t)$ at time t

$v_k(t)$: volatility of $P(t, t_k)$ at time t

δ_k : $t_{k+1} - t_k$

Volatility Structure

We assume a stationary volatility structure where the volatility of $F_k(t)$ depends only on the number of accrual periods between the next reset date and t_k [i.e., it is a function only of $k - m(t)$]

In Theory the Λ 's can be determined from Cap Prices

Define Λ_i as the volatility of $F_k(t)$ when $k - m(t) = i$

If σ_k is the volatility for the (t_k, t_{k+1}) caplet If the model provides a perfect fit to cap prices we must have

$$\sigma_k^2 t_k = \sum_{i=1}^k \Lambda_{k-i}^2 \delta_{i-1}$$

This allows the Λ 's to be determined inductively

Example 31.1 (Page 716)

- If Black volatilities for the first three caplets are 24%, 22%, and 20%, then

$$\Lambda_0 = 24.00\%$$

$$\Lambda_1 = 19.80\%$$

$$\Lambda_2 = 15.23\%$$

Example 31.2 (Page 716)

n	1	2	3	4	5
$\sigma_n(\%)$	15.50	18.25	17.91	17.74	17.27
$\Lambda_{n-1}(\%)$	15.50	20.64	17.21	17.22	15.25
n	6	7	8	9	10
$\sigma_n(\%)$	16.79	16.30	16.01	15.76	15.54
$\Lambda_{n-1}(\%)$	14.15	12.98	13.81	13.60	13.40

The Process for F_k in a One-Factor LIBOR Market Model

$$dF_k = \dots + \Lambda_{k-m(t)} F_k dz$$

The drift depends on the world chosen

In a world that is forward risk - neutral

with respect to $P(t, t_{i+1})$, the drift is zero

Rolling Forward Risk-Neutrality (Equation 31.12, page 717)

It is often convenient to choose a world that is always FRN wrt a bond maturing at the next reset date. In this case, we can discount from t_{i+1} to t_i at the δ_i rate observed at time t_i . The

process for F_k is

$$\frac{dF_k}{F_k} = \sum_{j=m(t)}^i \frac{\delta_i F_i \Lambda_{i-m(t)} \Lambda_{k-m(t)}}{1 + \delta_i F_i} dt + \Lambda_{k-m(t)} dz$$

The LIBOR Market Model and HJM

In the limit as the time between resets tends to zero, the LIBOR market model with rolling forward risk neutrality becomes the HJM model in the traditional risk-neutral world

Monte Carlo Implementation of LMM Model

(Equation 31.14, page 717)

We assume no change to the drift between reset dates so that

$$F_k(t_{j+1}) = F_k(t_j) \exp \left[\left(\sum_{i=k}^j \frac{\delta_i F_i(t_j) \Lambda_{i-j} \Lambda_{k-j}}{1 + \delta_j L_j} - \frac{\Lambda_{k-j}^2}{2} \right) \delta_k + \Lambda_{k-j} \varepsilon \sqrt{\delta_j} \right]$$

Multifactor Versions of LMM

- LMM can be extended so that there are several components to the volatility
- A factor analysis can be used to determine how the volatility of F_k is split into components

Ratchet Caps, Sticky Caps, and Flexi Caps

- A plain vanilla cap depends only on one forward rate. Its price is not dependent on the number of factors.
- Ratchet caps, sticky caps, and flexi caps depend on the joint distribution of two or more forward rates. Their prices tend to increase with the number of factors

Valuing European Options in the LIBOR Market Model

There is an analytic approximation that can be used to value European swap options in the LIBOR market model. See equations 31.18 and 31.19 on page 721

Calibrating the LIBOR Market Model

- In theory the LMM can be exactly calibrated to cap prices as described earlier
- In practice we proceed as for short rate models to minimize a function of the form

$$\sum_{i=1}^n (U_i - V_i)^2 + P$$

where U_i is the market price of the i th calibrating instrument, V_i is the model price of the i th calibrating instrument and P is a function that penalizes big changes or curvature in a and σ

Types of Mortgage-Backed Securities (MBSs)

- Pass-Through
- Collateralized Mortgage Obligation (CMO)
- Interest Only (IO)
- Principal Only (PO)

Option-Adjusted Spread (OAS)

- To calculate the OAS for an interest rate derivative we value it assuming that the initial yield curve is the Treasury curve + a spread
- We use an iterative procedure to calculate the spread that makes the derivative's model price = market price.
- This spread is the OAS.